Damiers de phase diophantiens
Pour l’Exoplanétologie (future)

**Daniel Rouan**, Damien Pickel, Jean-Michel Reess, Olivier Dupuis – LESIA
Didier Pelat – LUTH
Fanny Chemla, Mathieu Cohen – GEPI

Observatoire de Paris
Detecting an exoplanet?

![Graph showing the Sun and other celestial bodies with annotations](image1)

![Image of a solar system with labeled planets](image2)

![Image of a close-up view of a celestial body with labeled points A and B](image3)
In the real world, the planet is $10^5$ times weaker than here and the stellar light spreads much beyond the planet.

Where is the planet?

Detecting directly an exoplanet is TERRIBLY difficult!!!
Direct detection and characterization of a planet is thus a hard task

Two avenues:

- In visible: one telescope + adaptive optics + coronagraphy
- In infrared: multi-telescopes + space + nulling interferometry
A transparent mask

4 quadrants phase mask coronagraph
Bracewell-type nulling interferometer

Success depends on several challenges, one being the *achromatic* phase shift within the bandpass (typically one to two octave).
Life on exoplanet: tracers in the mid-IR

- Signatures of three key components: H$_2$O, O$_3$, CO$_2$
- Broad range of $\nu$: from 6 to 18 $\mu$m
- Detecting those bands: goal of a future Darwin-type space project

Observed spectra

D. Rouan - LESIA - JRIOA - Villetaneuse 10/07/13
Diophantine optics

- **Diophantus of Alexandria**: Greek mathematician, known as the « father of algebra »
- He studied polynomial equations with integer coefficients and integer solutions such as 
  \[(x-1)(x-2) = x^2 - 3x + 2 = 0\]
  called diophantine equation
- The most famous one: the Egyptian triangle \[5^2 = 4^2 + 3^2\]
- Optics and power of integers:
  - Constructive or destructive interferences = optical path differences multiple integer (odd or even) of \(\frac{\lambda}{2}\)
  - Complex amplitude = highly non-linear function of the opd
  → Taylor development implies powers of integers
- Diophantine optics = exploitation in optics of some remarkable algebraic relations between powers of integers
A few examples

- Avoiding long delay lines in space interferometers using fixed mirrors
- Obtaining deep and flat nulling function $\theta^n$ in multi-telescopes nulling interferometers
- Improving achromatism in phase mask coronagraph

D. Rouan - LESIA - JRIOA - Villetaneuse 10/07/13
Avoiding long delay lines

- Consider a 3-telescopes nulling interferometer in space
- How recombining the beams on one of the spacecraft carrying a telescope, without long delay lines which would be prohibitive in space?
- Use relay mirrors and diophantine relations between spacecraft distances

Optical path = $4+4+5 = 3+5+5 = 13$

Optical path = 13

Optical path = 18

D. Rouan - LESIA - JRIOA - Villetaneuse 10/07/13
Deep nulling

- In a Bracewell nulling interferometer, the stellar disk is resolved → leaks of light
- To obtain a deep and flat nulling function in a multi-telescopes interferometer
  - distribute the telescopes with a $\pi$ phase shift according to the Prouhet-Thué-Morse series 0110100110010110...
  - nulling varies then as $\theta^n$ with n as high as wished
- The coefficients of the Taylor development of the complex amplitude vanish thanks to diophantine relations between sums of powers of $2^l$ first integers

$$0 \ 1 \ 1 \ 0 \ 1 \ 0 \ 0 \ 1 \ \ldots$$
$$1 \ 2 \ 3 \ 4 \ 5 \ 6 \ 7 \ 8 \ \ldots \ 2^l \quad \sum n_k^p = \sum m_k^p \quad \text{for all } \ p < L-1$$

$$\theta = j \varphi \sum (m_k - n_k) - \varphi^2 / 2 \sum (m_k^2 - n_k^2) - j \varphi^3 / 6 \sum (m_k^3 - n_k^3) + \ldots$$

D. Rouan - LESIA - JRIOA - Villetaneuse 10/07/13
Achromatism of phase mask coronagraph

- Improving achromatism in phase mask coronagraph
- Play with combinations of quadrant thickness to cancel the first terms of the Taylor development of the complex amplitude $\mathcal{A} = \exp(j k \mathcal{R})$ where $\mathcal{R} = k \mathcal{Q}$.
  - 4QPMC: 2nd order with steps = [0, 1, 2, 1] $a \approx (\mathcal{Q})^2$
  - 8QPMC: 3rd order with steps = [1, 8, 3, 6, 2, 7, 2, 7] $a \approx (\mathcal{Q})^3$
- A 4QPMC component just manufactured: to be tested soon

D. Rouan - LESIA - JRIOA - Villetaneuse 10/07/13
Achromatic phase shift in a nulling interferometer

The (diophantine) chessboard phase shifter
A new concept of achromatic phase shifter: the achromatic chessboard


Based on a single optical device and some unforeseen application of diophantine equations

The wavefront is divided into many sub-pupils with two “chessboards” of phase-shifting cells, each producing an opd: an even or odd multiple of $\lambda/2$

The proper distribution of opd produces the quasi-achromatization

Main potential interest: makes the arms of the interferometer fully symmetric

\[
P \text{ (even pupil)} \quad \text{opd} = 2k \times \lambda_0/2
\]

\[
Q \text{ (odd pupil)} \quad \text{opd} = (2k + 1) \times \lambda_0/2
\]
Opd distribution: some mathematics...

- If $z = (-1)^{\lambda_0 / \lambda} \equiv e^{i\pi\lambda_0 / \lambda}$
- The cell with $\text{opd} = k \lambda_0 / 2$ has a complex amplitude $z^k$
- For a basic Bracewell (one cell per chessboard) the amplitude is $\Lambda = 1 + z$
  That is $\lambda = \lambda_0$ induces a root of order one on $\Lambda$.
- To obtain a flat $\Lambda$ around $\lambda_0$ let’s produce a multiple root: $\Lambda = (1 + z)^n$
  - The higher $n$ the deeper the nulling vs $\lambda$ around $\lambda_0$
This generates the number of cells of a given phase shift according to the binomial coefficients.

- E.g. for $n = 3$, we get $(1 + z)^3 = 1 + 3z + 3z^2 + z^3 = 1 + z + z + z + z^2 + z^2 + z^3$
- producing 1 opd of $0$, 3 of $\frac{\lambda_0}{2}$, 3 of $\frac{2\lambda_0}{2}$ and 1 of $\frac{3\lambda_0}{2}$
- And for a $2 \times 32 \times 32$ chessboard: $(1 + z)^{11}$
**x,y-distribution of the steps**

- \( P_r \) and \( Q_r \) = physical arrangement of the phase shifters at order
- The cells are placed in such a way that \( P_r - Q_r \) is a finite difference differential operator of high order
- Light in the focal plane is rejected outside: this improves the nulling
- One can achieve this objective (Pelat, Rouan, Pickel, 2010) with the following iterative arrangement:

\[
P_{r+1} = \begin{bmatrix} Q_r + 1 & P_n + 2 \\ P_r & Q_r + 1 \end{bmatrix}, \quad Q_{r+1} = \begin{bmatrix} P_r + 1 & Q_n + 2 \\ Q_r & P_r + 1 \end{bmatrix}
\]
Performances of achromatization

- Theoretical estimate:
- Absolute max bandpass: $\Delta \lambda = \frac{2}{3} \lambda_0 - 2\lambda_0 \Rightarrow$ a factor 3 in $\Delta \lambda$
- if 64x64 cells: $\Delta \lambda = (.65 \lambda_0 - 1.3 \lambda_0) \Rightarrow$ one octave.
- Darwin specs (6 – 18 µm) achievable with two components

D. Rouan - LESIA - JRIOA - Villetaneuse 10/07/13
The DAMNED* test bench

- Uses 2 off-axis parabola, a chessboard mask, a single-mode fiber
- Simulates two contiguous telescopes recombined in a Fizeau scheme
- The single-mode fiber is essential in the Fizeau scheme: it allows to sum the anti-symmetric amplitude and thus to make the nulling effective
- Measurement: x-y scanning of the focus with the fiber optics head

* Dual Achromatic Mask Nulling Experimental Demonstrator

D. Rouan - LESIA - JRIOA - Villetaneuse 10/07/13
The DAMNED* test bench

D. Rouan - LESIA - JRIOA - Villetaneuse 10/07/13
Transmissive chessboard mask

- Manufactured by GEPI – Obs de Paris
- \(2 \times 8 \times 8\) cells of 600 µm size in amorphous silica, using Reactive Ion Etching
- Result: typical nulling factor: \(3-7 \times 10^{-3}\) for broad-band filters;
- Quasi-achromatism is indeed obtained: \(8 \times 10^{-3}\) in the range 460-840 nm
- The medium nulling factor does agree with simulations using the actual mask cell’s thickness: performances are limited by the mask’s step accuracy
- Need for a better accuracy of steps thickness

D. Rouan - LESIA - JRIOA - Villetaneuse 10/07/13
Use of a segmented AO mirror

- Phase chessboard synthesized using a segmented deformable mirror:
  - free choice of the central wavelength
  - fine control of each cell’s opd
  - versatile way to change the XY distribution
  - open the door to modulation

- Choice of a segmented Boston µ-machine 12 x 12 electrostatic mirror

- Optical scheme adapted to work in reflection

- Control of flatness using phase contrast
DM control using strioscopy

Control of flatness using phase contrast (strioscopy): OK

Accuracy: typically 2-3 nm

Step by step procedure to flatten the DM
Performances w segmented AO mirror

- The parabola was no longer suited for the larger PSF (small DM) ➔ direct image of the *coffee bean* ➔ performance assessment less accurate
- Another method: scanning by the DM at a fixed $\lambda$ of the source (laser)
- Performances similar to the transmissive chessboard while order is lower ($2 \times 4 \times 4$)
Present work:
- improvement of the piston accuracy
- better optical design
- spectroscopy for simultaneous chromatic performances

Future work: **modulation** between different nulling configurations to measure possible biases

Extrapolation to mid-IR:
If the accuracy on piston is the same at 10 µm, a null depth of $2 \times 10^{-6}$ would be reached on a low order $(2 \times 8 \times 8)$ chessboard: **within the specifications**!
**Conclusion - summary**

- *Diophantine optics*: another way of thinking problems in optics
- In some cases it may bring a genuine solution
- Principle of the *achromatic chessboard phase shifter* demonstrated in the lab in reflection and transmission
- Performances and mode of operation using a *segmented deformable mirror* demonstrated: brings a clear asset

*Diophantine optics also brings the best solution for sharing a salami between several fellow guests....*
Thanks for your attention